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# Simple stochastic models showing strong anomalous diffusion

K.H. Andersen[1,a], P. Castiglione[1,2], A. Mazzino[3], and A. Vulpiani[1]

[1] Dipartimento di Fisica, and INFM, Università "La Sapienza", P.le A. Moro 2, 00185 Roma, Italy
[2] Laboratoire de Physique Statistique, École Normale Supérieure, 24 rue Lhomond, 75231 Paris Cedex 05, France
[3] INFM–Dipartimento di Fisica, Università di Genova, 16146 Genova, Italy

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**Abstract.** We show that *strong* anomalous diffusion, *i.e.* $\langle|x(t)|^q\rangle \sim t^{q\nu(q)}$ where $q\nu(q)$ is a nonlinear function of $q$, is a generic phenomenon within a class of generalized continuous-time random walks. For such class of systems it is possible to compute analytically $\nu(2n)$ where $n$ is an integer number. The presence of strong anomalous diffusion implies that the data collapse of the probability density function $P(x,t) = t^{-\nu}F(x/t^{\nu})$ cannot hold, a part (sometimes) in the limit of very small $x/t^{\nu}$, now $\nu = \lim_{q\to 0}\nu(q)$. Moreover the comparison with previous numerical results shows that the shape of $F(x/t^{\nu})$ is not universal, *i.e.*, one can have systems with the same $\nu$ but different $F$.

**PACS.** 05.45.-a Nonlinear dynamics and nonlinear dynamical systems – 05.60.-k Transport processes

## 1 Introduction

Anomalous diffusion, *i.e.*, when the scaling of the moments of the position $x(t)$ is $\langle x^2(t)\rangle \sim t^{2\nu}$ with $\nu > 1/2$, has been observed in a rather wide class of dynamical systems, *e.g.*, intermittent maps [1,2], 2D symplectic maps [3–5] and random velocity field [6–8] as well in 2D time-dependent flow [9] and Hamiltonian systems (*e.g.* the egg-crate potential) [10,11]. In highly nontrivial systems, as those described in [9,10], the existence of anomalous diffusion has been established only numerically. On the other hand, for random shears it is possible to give an analytical criterium both for the existence of anomalous diffusion and for the computation of $\nu$ [12]. As far as we know, the simplest nontrivial system showing anomalous diffusion is the continuous-time random walk (CTRW), sometimes also called Lévy walk. The CTRW is entirely specified by the probability density function (pdf) $\psi(r,\tau)$ to move a distance $r$ in a time $\tau$ in a single motion event. Let us assume, as in [13–15],

$$\psi(r,\tau) = P(\tau)\,P(r \mid \tau), \tag{1}$$

where $P(\tau)$ is the pdf of having a flight of duration $\tau$ and $P(r \mid \tau)$ is the conditional pdf of a displacement $r$ given the flight time $\tau$. The cases corresponding to $P(\tau) \sim \tau^{-g}$ and $P(r \mid \tau) = \delta(\mid r \mid - \tau^{\alpha})/2$ can be treated analytically [13–15].

If the scaling of all moments can be described by just one exponent, *i.e.* $\langle x^{2n}(t)\rangle \sim t^{2n\nu}$, a collapse of the pdf's at different times is obtained exploiting the rescaling [9]

$$P(x,t) = t^{-\nu}F(x/t^{\nu})\ . \tag{2}$$

Then it also becomes clear that the value of $\nu$, in general, does not completely characterize the statistical properties of the diffusion process, as the function $F(\xi)$ needs to be specified.

In many cases, the use of just one exponent is not enough to describe all the moments, *i.e.*, we have the so-called *strong* anomalous diffusion [9,10]. For which $\langle|x(t)|^q\rangle \sim t^{q\nu(q)}$, where $q\nu(q)$ is a nonlinear function of $q$. The existence of a non-unique scaling exponent implies the failure of the data collapse for the pdf in the form given by equation (2). The best known case of a process showing strong anomalous diffusion is the advection of a passive scalar by a turbulent velocity field. In many cases [9] it has been observed that the function $q\nu(q)$ is piecewise linear, *i.e.*,

$$q\,\nu(q) \simeq \begin{cases} \nu_1 q & q < q_c \\ q - c & q > q_c. \end{cases} \tag{3}$$

This is basically due to the existence of two mechanisms: a *weak* (*i.e.*, with a unique exponent $\nu_1 > 1/2$) anomalous diffusion for the typical events, and a ballistic transport for the rare excursions (*i.e.*, excursions much larger than $x_{\rm typ}(t) \equiv \exp\langle\ln x(t)\rangle$). The behavior (3) suggests the validity of the data collapse (2) for the pdf core, *i.e.* $x/t^{\nu_1}$ not too large, and two peaks at $\mid x \mid\sim t$, *i.e.*, the footprint of ballistic events. The cases where strong anomalous diffusion for which (3) holds are relatively simple and surely

[a] *Present address:* Department of Hydrodynamics and Water Resources, Technical University of Denmark, Building 115, 2800 Lyngby, Denmark.
e-mail: `ken@isva.dtu.dk`

different from the cases of the relative dispersion in the fully developed turbulence [17].

By using only elementary techniques, in this paper we show that the bi-linear behavior for the scaling of the moments (3), which is present in the special case of the CTRW commonly found in the literature, *i.e.* $P(r \mid \tau) = \delta(\mid r \mid -\tau^{\alpha})/2$, does not hold in the general case. To show this point, we shall consider a generalized CTRW of the form:

$$P(r \sim \tau^{1+h} \mid \tau) \sim \tau^{-S(h)}. \tag{4}$$

The inspiration for this choice comes from the multifractal description of turbulence [19].

The paper is organized as follows. In Section 2.1 we present the "standard" CTRW model. We then present a simple method to find the scaling for the even order moments (Sect. 2.2). In Section 2.3 the model is generalized, and the same method is again applied to find the scaling. Numerical analysis to corroborate the analytical results of the general model are presented in Section 3 together with some discussions related to the shape of $P(x,t)$. Discussions and conclusions can finally be found in Section 4.

## 2 CTRW models

Anomalous diffusion occurs when some, or all, of the hypothesis of the central limit theorem break down. More specifically, the system has to violate at least one of the two following conditions:

1. Finite variance of the velocity.
2. Fast enough decay of the auto-correlation function of the Lagrangian velocities.

The paradigmatic model for anomalous diffusion, namely the Lévy flights [18] violate the first condition. In the one dimensional case a Lévy flight corresponds to the evolution in discrete time

$$x(t_{i+1}) = x(t_i) + v_i \Delta t \tag{5}$$

of the particle position $x$ with $t_{i+1} = t_i + \Delta t$ and $v_i$ being independent stochastic variables identically distributed according to a Lévy-stable distribution such that

$$P(v) \sim v^{-g} \text{ for large } v \tag{6}$$

where $1 < g \leq 3$. It is easy to show that $\langle x^2 \rangle = \infty$ for $g < 3$ and that this stochastic process shows anomalous diffusion being $x_{\text{typ}} \sim t^{1/(g-1)} > t^{1/2}$.

### 2.1 The "standard" CTRW model

The existence of an infinite variance is not very pleasing from a physical point of view. This has lead to the introduction of the CTRW (also called Lévy walks). The idea is to relax the condition of a fixed, discrete time step in such a way that the process still has anomalous diffusion, but finite variance of the velocity.

Firstly, we introduce the particle trajectory

$$x(t) = x(t - \tau_i) + v_i \tau_i \tag{7}$$

where $x(t)$ denotes the position of the particle at time $t$. During the random intervals $\tau_i$, the particles move a distance $r_i$ with constant random velocity $v_i$ independent of $\tau_i$. After each interval they choose new random values for $\tau_i$ and $v_i$.

The relevant quantity to characterize the motion of the particle is the pdf $\psi(r,\tau)$ of having a displacement $r$ in time $\tau$ in a single motion event. This pdf is chosen in the form (1). In the simple case where $v_i = \pm v$ we have

$$P(r \mid \tau) = \frac{1}{2}\delta(\mid r \mid - v\tau) \ . \tag{8}$$

Taking

$$P(\tau) \sim \tau^{-g} \quad \text{for large } \tau \tag{9}$$

we can determine the pdf $P(x,t)$ to be in $x$ at time $t$ [20,21]. Actually, by introducing the probability density $\Psi(x,t)$ to pass at location $x$ at time $t$ in a single motion event (and not necessarily to stop at $x$)

$$\begin{aligned} \Psi(x,t) &= P(x \mid t) \int_t^{\infty} \mathrm{d}\tau \int_{|x|}^{\infty} \mathrm{d}r\ \psi(r,\tau) \\ &= \frac{1}{2}\delta(\mid x \mid - vt) \int_t^{\infty} \mathrm{d}\tau\ P(\tau) \end{aligned} \tag{10}$$

the pdf $P(x,t)$ can be written in the following way

$$\begin{aligned} P(x,t) = \Psi(x,t) &+ \int_{-\infty}^{\infty} \mathrm{d}x' \\ &\times \int_0^t \mathrm{d}\tau\ \psi(x',\tau)\Psi(x-x',t-\tau) + ... \end{aligned} \tag{11}$$

the first term denotes the probability density to reach the position $x$ at time $t$ in a single motion event, the second term is the probability density to reach $x$ at time $t$ with one stop in $x'$ and so on to include all the combinations of motion events. In the Fourier-Laplace space ($x \to k, t \to u$) the series in equation (11) assumes the closed form

$$\hat{P}(k,u) = \frac{\hat{\Psi}(k,u)}{1 - \hat{\psi}(k,u)} \tag{12}$$

and the behavior of $\langle x(t)^2 \rangle$ can be calculated analytically by using the relation

$$\langle x(t)^2 \rangle = -\mathcal{L}^{-1}\left[\frac{\partial^2}{\partial^2 k}\hat{P}(k,u) \mid_{k=0}\right],$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform ($u \to t$) [20,21].

The results is that

$$\langle x(t)^2 \rangle \sim \begin{cases} t^2 & 1 < g < 2 \quad \text{ballistic motion} \\ t^{4-g} & 2 < g < 3 \quad \text{anomalous diffusion} \\ t & 3 \ < g \quad \text{normal diffusion.} \end{cases} \tag{13}$$

Thus, enhanced anomalous diffusion occurs for $2<g<3$. For the moments of small order, which describe the core of the pdf, it has been shown that the asymptotic behavior gives $\langle |x(t)|^q\rangle \sim t^{q\nu}$ with $\nu = 1/(g-1)$ [20,21]. Thus the core of the pdf can be scaled as in (2) using $\nu$. The ballistic motions which are responsible for the different scaling of higher order moments, show up as wings on the pdf, which does not scale using $\nu$.

The previous approach can be generalized [16] to the case where

$$\begin{aligned} P(r \mid t) &= \frac{1}{2}\delta(\mid r \mid - \tau^{\alpha}), \quad v = \pm\tau^{\alpha-1}, \\ P(\tau) &\sim \tau^{-g} \ \text{ with } \ g > 1. \end{aligned} \tag{14}$$

In addition one can treat more complicated situations [10,15] by considering that the particle can move ballistically but it can be also trapped in some structures as vortices [22] or chaotic islands (standard map or "egg-crate" potential) [21,23].

## 2.2 Finding the scaling of the moments

The usual method used in [16,20,21] to find the scaling of the moments in the simple CTRW model is not elementary. We now present an alternative easier way to calculate the displacement moments $\langle x^q(t)\rangle$ and thus to characterize the anomalous diffusion.

Let us consider a particle moving ballistically with velocity $v_i$ during the interval times $\tau_i$. The velocities $v_i$ are identically distributed, independent, random variables assuming the values $\pm 1$ alternatively. The intervals times $\tau_i$ are identically distributed, independent, random variables and assuming the value $\tau$ with probability

$$P(\tau) \sim \tau^{-g} \ \text{ with } \ g > 1 \ \text{ and } \ \tau \in [t_{\min}, T] \tag{15}$$

where $t_{\min}$ and $T$ are the lowest and highest cutoffs, respectively. The reason for which we need to introduce such cutoffs will be clear later. The particle position at the time $t$ can be written as

$$\begin{aligned} x(t) &= \sum_{i=1}^{n} v_i\tau_i + \, v_{n+1}\epsilon_{n+1} \\ \text{with } \ \epsilon_{n+1} &= t - \sum_{i=1}^{n} \tau_i \end{aligned} \tag{16}$$

$n$ being the (random) integer value for which $t_n \le t$ and $t_{n+1} > t$. The total time $t$ can be rewritten as $t = \sum_{i=1}^{n} \tau_i + \epsilon_{n+1}$. Denoting by $N \equiv \langle n\rangle$ the average value of the number of time steps necessary to reach the time $t$, one has for large times $t \simeq N\langle\tau\rangle$ and thus $x(t) = \sum_{i=1}^{n} v_i\tau_i$.

From simple considerations related to the symmetry of the velocity pdf under the transformation $v \mapsto -v$, it immediately follows that the odd-order moments $\langle x(t)^q\rangle$ are trivially zero. Conversely, even-order moments are nonzero and can be evaluated exploiting the following properties: $\langle v_i\tau_j\rangle = 0$, $\langle v_i v_j\rangle \propto \delta_{ij}, \langle\tau_i\tau_j\rangle \propto \delta_{ij}$. For times large enough, the mean squared displacement thus reads:

$$\begin{aligned} \left\langle x(t)^2 \right\rangle &= \left\langle \sum_{i=1}^{n}\sum_{j=1}^{n} (v_i\,\tau_i)\,(v_j\,\tau_j) \right\rangle \\ &= N\left\langle (v\,\tau)^2\right\rangle . \end{aligned} \tag{17}$$

Similarly, for the fourth and sixth order moments the limit of large times yields

$$\begin{aligned} \left\langle x^4(t)\right\rangle &= N \ \left\langle (v\tau)^4\right\rangle + 3\ N^2 \ \left\langle (v\tau)^2\right\rangle^2 \\ \left\langle x^6(t)\right\rangle &= N \ \left\langle (v\tau)^6\right\rangle + 15\ N^2 \ \left\langle (v\tau)^2\right\rangle \left\langle (v\tau)^4\right\rangle \\ &\quad + 15\ N^3 \ \left\langle (v\tau)^2\right\rangle^3 \end{aligned} \tag{18}$$

and so on.

Our attention being focused on the behavior of $\langle x(t)^q\rangle$ as a function of $t$, we make the substitution $N = t/\langle\tau\rangle$ in previous expressions (17) and (18) and again exploit the facts that $v_i$ and $\tau_i$ are uncorrelated and $\left\langle v_i^2\right\rangle = 1$ to get:

$$\begin{aligned} \left\langle x^2(t)\right\rangle &= \frac{t}{\langle\tau\rangle}\left\langle\tau^2\right\rangle \\ \left\langle x^4(t)\right\rangle &= \frac{t}{\langle\tau\rangle}\left\langle\tau^4\right\rangle + 3\left(\frac{t}{\langle\tau\rangle}\right)^2\left\langle\tau^2\right\rangle^2 \\ \left\langle x^6(t)\right\rangle &= \frac{t}{\langle\tau\rangle}\left\langle\tau^6\right\rangle + 15\left(\frac{t}{\langle\tau\rangle}\right)^2\left\langle\tau^2\right\rangle\left\langle\tau^4\right\rangle \\ &\quad + 15\left(\frac{t}{\langle\tau\rangle}\right)^3\left\langle\tau^2\right\rangle^3 . \end{aligned} \tag{19}$$

For times $t \gg T$ the leading term for $\left\langle x^{2n}(t)\right\rangle$ is that one proportional to $t^n$, therefore one has:

$$\left\langle x(t)^{2n}\right\rangle \propto t^n \left(\frac{\left\langle\tau^2\right\rangle}{\langle\tau\rangle}\right)^n \quad \text{where } n \text{ is integer} \tag{20}$$

which is just ordinary diffusive behaviour. Diffusive behavior in such limit is actually expected from general considerations. Indeed, when $t \gg T$ the particle position at the time $t$ can be rearranged in the form of a sum of almost independent displacements. If the number of the latter is large enough, central limit arguments apply, with the immediate consequence is that particles undergo diffusive motion. Such result can also be rigorously proved exploiting multiscale perturbative expansions in the (small) parameter $T/t$ as done, *e.g.*, in reference [12].

In the opposite regime, where $t \ll T$, the system is strongly correlated, central limit arguments does not apply, and the final result is that non-diffusive (*i.e.* anomalous) regimes can occur where $\langle x^q(t)\rangle \sim t^{q\nu(q)}$ with $\nu(q) \neq 1/2$.

The possible emergence of anomalous behaviors in the limit $t \to \infty$ can be investigated by looking at the dependence of moments on the cutoff $T$. For times shorter than $T$ they behave as $\langle x^q(t)\rangle \sim t^{q\nu(q)}$, but around $t \sim T$ the moments have a crossover to diffusive behaviour, $\langle x^q(t)\rangle \sim (t/\langle\tau\rangle)^{q/2}\left\langle\tau^2\right\rangle^{q/2}$.

By matching the two different regimes at $t = T$, and using the results $\langle \tau^q \rangle \sim T^{-g+q+1}$ for $-g+q+1 > 0$ and $\langle \tau^q \rangle = O(1)$ for $-g+q+1 < 0$, the following expressions for $q\nu(q)$ are found as a function of the exponent $g$:

$$
\begin{array}{lll}
g \in (1,2] & q\nu(q) = q & q = 2,4,6,... \\[2ex]
g \in (2,3] & q\nu(q) = q+2-g & q = 2,4,6,... \\[2ex]
g \in (3,4] & q\nu(q) = q/2 & q = 2 \\
 & q\nu(q) = q+2-g & q = 4,6,8,...
\end{array}
\tag{21}
$$

and so on for higher values of $g$.

From (21) it follows that the anomalous diffusion phase takes place for higher and higher moments as $g$ increases. For $q$ large enough one has $q\nu(q) = q + \text{const.}$ and noting that $2\nu(2) \neq 2$ one can conclude that $\nu(q)$ can not be constant and therefore a strong anomalous diffusion is present.

The matching argument had been used in reference [12] in the context of the multiscale method for the anomalous diffusion in random shear flows [6].

### 2.3 Generalized CTRW model

We now present a generalization of the previous model showing a *strong* anomalous diffusion regime characterized by a non-piecewise linear behavior as a function of the order $q$.

As in the previous case the particle moves ballistically with random velocity $v_i$ during the random interval times $\tau_i = t_{i+1} - t_i$. We assume that $\tau_i$ has the same pdf as in (15) but now the velocities $v_i$ assume the value $\pm\tau_i^h$ where $h$ is a random positive variable conditioned on $\tau_i$. Specifically, the conditional pdf $P(h \mid \tau)$ is

$$P(h \mid \tau) \propto \tau^{-S(h)} \tag{22}$$

where $S(h)$ is a positive smooth concave function. This has the effect of giving a larger variance to the velocity, the larger the time $\tau_i$. The function $S(h)$ can be taylored to a special need, *i.e.*, to mimic the intermittent turbulent velocity field. Here we just use a generic, simple function $S(h) = h^2/(2\sigma^2)$, as we are mainly interested in the generic properties of the model.

The moments $\left\langle (v\,\tau)^{2q} \right\rangle$ can now be calculated:

$$\left\langle (v\,\tau)^{2q} \right\rangle = \int_0^{+\infty} \mathrm{d}h \int_{t_{\min}}^{T} \mathrm{d}\tau\, (v\tau)^{2q} P(\tau) P(h \mid \tau) \tag{23}$$

$$\sim \int_{t_{\min}}^{T} \mathrm{d}\tau\, \tau^{-g+2q} \int_0^{+\infty} \mathrm{d}h\, \tau^{2q\,h - S(h)} \tag{24}$$

$$= \int_{t_{\min}}^{T} \mathrm{d}\tau\, \tau^{-g+2q+y(2q)} \sim T^{-g+2q+y(2q)+1} \tag{25}$$

where the integral $\int \mathrm{d}h\, \tau^{2q\,h - S(h)}$ has been evaluated exploiting the steepest descent method and we have defined

$$y(2q) \equiv \max_h [2q\,h - S(h)]. \tag{26}$$

Now using the specific shape of $S(h)$, expression (26) takes the form:

$$y(2q) = q^2 y(2) \;\text{ with }\; y(2) = 2\,\sigma^2. \tag{27}$$

Exploiting the same matching arguments as in Section 2.2, the following expressions for the exponents $q\nu(q)$ are obtained:

$$
\begin{array}{ll}
g \in (1,2] & q\nu(q) = \frac{q^2}{2}\sigma^2 + q \\
 & q = 2,4,6,... \\[2ex]
g \in (2, 3+y(2)] & q\nu(q) = \frac{q^2}{2}\sigma^2 + q + 2 - g \\
 & q = 2,4,6,... \\[2ex]
g \in (3+y(2), 4+y(4)] & q\nu(q) = q/2 \\
 & q = 2 \\
 & q\nu(q) = \frac{q^2}{2}\sigma^2 + q + 2 - g \\
 & q = 4,6,8,...
\end{array}
\tag{28}
$$

and so on, for higher values of $g$.

We now see that in the general case of the CTRW, $q\nu(q)$ is not just a bi-linear function, and the exact form depends upon the shape of $S(h)$. The generalization to an arbitrary shape of $S(h)$ is straightforward.

As far as we know it is not simple to obtain the results in equation (21) and equation (28) with the method discussed in Section 2.1.

## 3 Numerical simulations

We present results of numerical simulations of the general model defined by equations (15, 22). The goal of the numerical simulations has been to verify the validity of the theoretical expectations (21) and (28) and to study the pdf of $x(t)$ in more detail.

In practice the pdf for the length of the jumps $P(\tau)$ has to be supplemented with a lower cutoff. As it is the tail of the distribution which governs the scaling of the moments of $x(t)$ the results are independent upon how the cutoff is made. We used a hard cutoff at $\tau = 1$, such that $P(\tau) = 0$ for $\tau < 1$ and

$$P(\tau) = (g-1)\tau^{-g} \quad \textit{for} \quad \tau > 1\,. \tag{29}$$

The moments have been calculated as an ensemble average of many realizations of the process for times $0 < t < \tilde{T}$. For all the simulations presented here, $\tilde{T}$ has been set to $10^6$.

In Figure 1 the scaling of the second moment $\left\langle x^2(t) \right\rangle$ with $t$ for different values of $g$ and $\sigma = 0.2$ is shown.

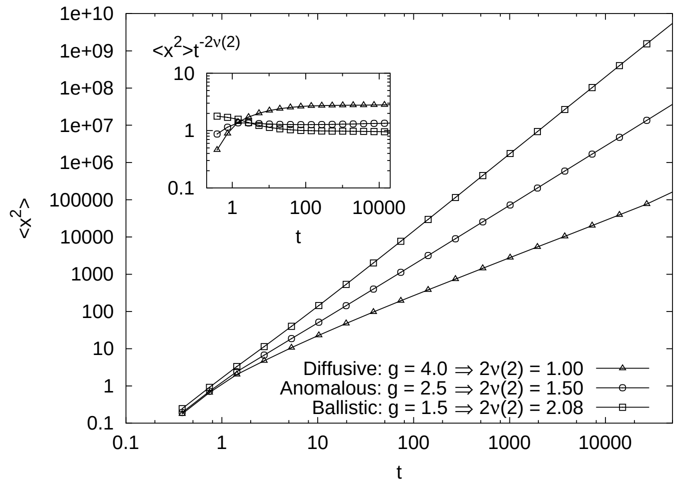


**Fig. 1.** The scaling of the displacement $\langle x^2(t)\rangle$ for three different values of $g$, each corresponding to diffusive behavior ($g = 4.0$), anomalous diffusion ($g = 2.5$) and ballistic motion ($g = 1.5$). The value of $\sigma$ is 0.2 for all the cases. In the inset the same is shown, but this time rescaled with the expected behavior $\langle x^2\rangle \propto t^{2\nu(2)}$, such that a scaling in accordance with the theoretical prediction corresponds to a horizontal line.

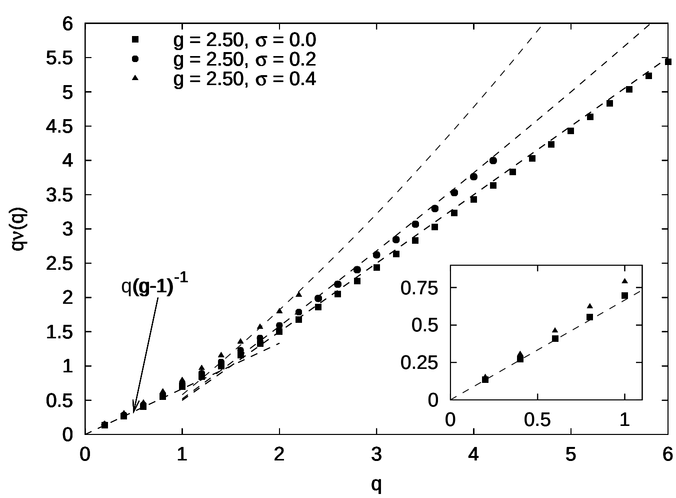


**Fig. 2.** The scaling of the higher order moments for three examples, with $g = 2.50$ and $\sigma = 0$, 0.2 and 0.4. The line originating from $q = 0$ corresponds to $q/(g-1)$. The other curves corresponds to $q\nu(q) = q + 2 - g + (q\sigma)^2/2$.

Examples have been chosen where diffusive, anomalous and ballistic behavior is expected. After an initial ballistic motion for short times, a transient towards scaling is taking place. For long times, clean scaling with an exponent corresponding to equations (21) is evident.

To show how the introduction of the velocity field from Section 2.3 influences the moments, we have calculated the higher order moments of $x(t)$ for $\sigma = 0$, 0.2 and 0.4 for $g = 2.5$ (Fig. 2). For the case with $\sigma = 0$ we see a clear bi-linear behavior as expected, with a cross over at $q_c \approx 1.5$. For low order moments ($q < 1$) the core of the pdf is important,

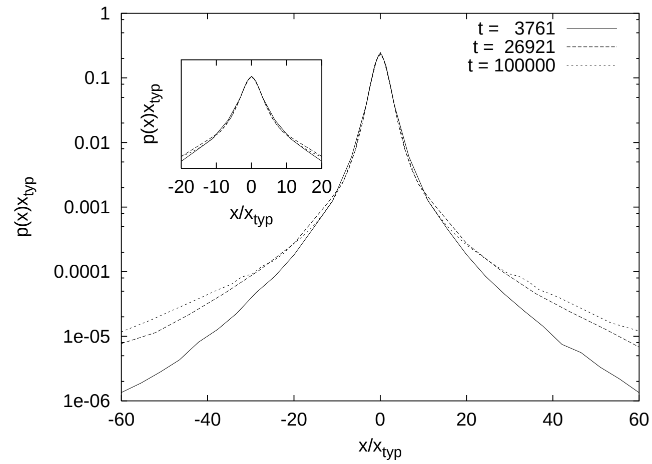


**Fig. 3.** The pdf of $x(t)$ at three different times rescaled according to equation (2) for a situation with $g = 2.5$ and $\sigma = 0.2$ corresponding to the anomalous regime. The pdf have been scaled with the typical value, $x_{\mathrm{typ}}(t)$.

and the behavior approaches $q/(g-1)$. As is seen in the inset, this behavior is only to be strictly valid in the limit $q \to 0$. The scaling of the moments changes smoothly into the prediction $q\nu(q) = q + 2 - g$ for $q \geq 2$. Using the fact that $q\nu(q)$ is a concave increasing function and, for $\sigma = 0$, the slope can not be larger than 1, one obtains that the prediction given by equation (21), obtained only for even order moments, surely is valid for any moment larger than 2. For the generalized model ($\sigma > 0$), the fluctuations become much stronger, and it was not possible to get a clear convergence for large orders. However, for $q = 2$ the prediction in equation (28) and the numerical results are in perfect agreement.

The characterization of the process by the scaling exponents of the moments can be seen as a way to probe the pdf $p(x, t)$ of the process. Roughly speaking, the low order structure functions characterize the core of the pdf, while higher orders characterize the tails. In the case of "ordinary" anomalous diffusion, the pdfs can be rescaled according to equation (2), due to the fact that the whole process can be characterized by just one scaling exponent. However for the bi-linear or the strong diffusive cases, no such renormalization can be done, as more than one exponent is needed to characterize the process. As all the cases seem to have the same limiting behavior for $q \to 0$, it might be possible to make a collapse of the core of the pdf. In Figure 3 the pdf for $g = 2.5$ and $\sigma = 0.2$ has been rescaled using the typical value $x_{\mathrm{typ}}(t) = \exp(\langle \ln(|x(t)|)\rangle$. As expected the core of the rescaled pdfs show a good overlap, while the tails diverge. For higher values of $\sigma$ the area of the core with overlap becomes smaller and smaller.

## 4 Discussions and conclusions

By using only elementary techniques, we have shown in this paper that the strong anomalous diffusion appears in CTRW. Our approach, in which one computes the even moments $\langle x(t)^q \rangle$ for a system with a cutoff $T$ on the pdf $P(\tau)$, and then the matching of the behaviors at $t \geq T$ and $t \leq T$, allows us to treat rather general cases. This is a relevant advantage with respect to the approach commonly used for CTRW which seems to us to be difficult to apply for more general cases. In the case of generalized CTRW, *i.e.* with equation (22), one has a nontrivial (non bi-linear) shape of $q\nu(q)$. This implies that the pdf $P(x,t)$ cannot be written in the form (2). On the other hand we found that the rescaling (2), with $\nu = \nu(0)$, is valid in a limited range of $x/x_{\rm typ}$. It is rather natural to wonder if, at least in this limited range, the function $F(\xi)$ is determined by the exponent $\nu(0)$. From previous works [24, 25] and the results discussed in Section 3, it seems to us that one has a negative answer: $\nu$ does not determine the function $F(\xi)$.

As the shape of the function describing the scaling of the moments looks similar to what one finds for the relative dispersion of a passive scalar by a turbulent velocity field, one might like to use the the CTRW model to describe this process. There is however some differences that has to be taken into consideration. In fully developed turbulence, a flight of duration $\tau$ can be considered as associated to an eddy of size $l \sim \tau^{3/2}$. In a process of relative diffusion, one typically has an increase of the relative distance with the time. This implies that a flight of duration $\tau$ is preferentially followed by another flight with a larger duration than the previous. The easiest way for a realistic description of relative dispersion in fully developed turbulence should thus consist in a further generalization of the CTRW where a Markov ingredient, *i.e.* a dependence of the flight duration at the time $t$ on the duration at the previous time, is introduced. It seems to us that this is a nontrivial task; a first attempt in this direction can be found in the recent work [26].

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